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# LIMIT LOADS OF EDGE-RESTRAINED SHALLOW SHELLS

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**Abstract**—Limit analysis of edge-restrained rigid, perfectly plastic shallow spherical shells under external pressure is considered. A numerical solution, based on the von Mises yield criterion, is shown to be in good agreement with existing approximate solutions based on the Tresca yield criterion.

## NOTATION

| | |
|---|---|
| $A$, $B$, $C$ | Arbitrary constants |
| $c$ | Shell parameter, equation (3) |
| $2H$ | Shell thickness |
| $h$ | Shell parameter, equation (3) |
| $M_r$, $M_\theta$ | Bending stress resultants in $r$ and $\theta$ direction, resp. |
| $m_r$, $m_\theta$ | Non-dimensional bending stress resultants |
| $M_0$ | Fully plastic bending moment |
| $N_r$, $N_\theta$ | Membrane stress resultants in $r$ and $\theta$ direction, resp. |
| $n_r$, $n_\theta$ | Non-dimensional membrane stress resultants |
| $N_0$ | Fully plastic membrane force |
| $p$ | Lateral uniform pressure |
| $P$ | non-dimensional pressure, equation (3) |
| $p^*$ | $\dfrac{p\pi R_1^2}{6\pi M_0}$, total non-dimensional pressure |
| $Q$ | Transverse shear resultant |
| $q$ | Non-dimensional transverse shear resultant |
| $r$, $z$ | coordinate directions |
| $R_1$, $R$ | Shell radii, Fig. 1 |
| $u$, $w$ | Shell displacements, Fig. 1 |
| $U$, $W$ | Non-dimensional shell displacements |
| $\alpha$ | Shell parameter, equation (3) |
| $\beta$ | Rotation of shell normal |
| $\xi$ | Non-dimensional radial shell coordinate |
| $\lambda$ | Flow parameter |
| $\sigma_0$ | Yield stress in simple tension. |

## INTRODUCTION

In a recent paper Jones and Ich[1] have considered the limit analysis of edge-restrained shallow shells under external pressure. They give complete solutions for various approximate yield surfaces based on the Tresca yield criterion. The resulting yield point load solutions are upper and lower bounds on the exact Tresca yield point loads. Apparently no exact solution for either Tresca or von Mises criterion exists for this problem.

It is the intent of this paper to give a solution of the title problem for a yield surface which is based on the von Mises criterion, and to compare the resulting yield point loads to the bounded solutions given in[1].

To indicate the usefulness of the method of analysis of the present problem, solutions for shells with unrestrained edges are briefly reviewed and compared with experimental data available.

## PROBLEM FORMULATION

The limit analysis of the shallow shell whose geometry is shown in Fig. 1. is considered. The shell material is assumed rigid, perfectly plastic and obeys the von Mises yield criterion. The loading consists of a uniformly distributed lateral pressure $p$ and the shell contour is hinge supported. It is further assumed that the shell contour is restrained to move in the inplane direction.

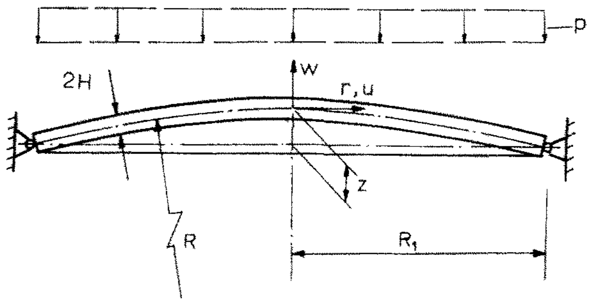


Fig. 1. Geometry of edge-restrained shell.

The analysis follows the usual rules of classical limit analysis[2], and is restricted to thin, shallow shells. These shells have the following geometrical restrictions:

$$c \leqq 0{\cdot}4 \tag{1}$$

and

$$\tfrac{1}{30} \leqq \alpha \leqq \tfrac{1}{15}. \tag{2}$$

The value of $c = 0{\cdot}4$ as a limiting "shallowness" parameter has been suggested in[3]. For larger values of $c$ the use of shallow shell equations may introduce errors of unknown magnitude into the analysis. For $\alpha > \frac{1}{15}$ the use of thin shell equations becomes questionable and for $\alpha < \frac{1}{30}$ stability considerations may be encountered. Equation (2) is therefore given here as an approximate range of $\alpha$ for which the present analysis is intended.

For convenience the following non-dimensional notation is introduced at this point:

$$\begin{aligned}
&n_r = \frac{N_r}{N_0}; \quad n_\theta = \frac{N_\theta}{N_0}; \quad m_r = \frac{M_r}{M_0}, \quad m_\theta = \frac{M_\theta}{M_0}\\
&q = \sqrt{3}\,\frac{Q}{N_0}; \quad N_0 = 2H\,\sigma_0; \quad M_0 = \sigma_0\,H^2, \quad \xi = \frac{r}{R}\\
&P = p\,\frac{R}{N_0}; \quad \alpha = \frac{2H}{2R_1}; \quad W = \frac{w}{R}\\
&U = \frac{u}{R}; \quad c = \frac{R_1}{R}; \quad h = \alpha\,\frac{c}{2}.
\end{aligned} \tag{3}$$

## GOVERNING EQUATIONS

The equations, describing the behaviour of the shell, are taken from[4]. The set of equations consists of the following:

(a) *The Equilibrium equations*

$$(\xi n_r)' - n_\theta = 0 \tag{4}$$

$$\xi n_r - \frac{1}{\sqrt{3}} q + \tfrac{1}{2}\xi P = 0 \tag{5}$$

$$(\xi m_r)' - m_\theta - \frac{1}{\sqrt{3}h}\xi q = 0. \tag{6}$$

(b) *The yield surface is that of Shapiro*[5], i.e.

$$n_r^2 + n_\theta^2 - n_r n_\theta + m_r^2 - m_r m_\theta + m_\theta^2 + q^2 - 1 = 0. \tag{7}$$

The positive direction of the stress resultants is shown in Fig. 2. The yield surface equation (7) is an approximation of the exact yield surface of Ilyushin[6] which is based on the von Mises criterion, "average" yielding and linear velocities through the shell wall. Note that this yield surface is valid only for thin shells, where the transverse normal shear stress $q$ is small. For these shells the boundary conditions on $q$ at the shell surface are approximately satisfied.

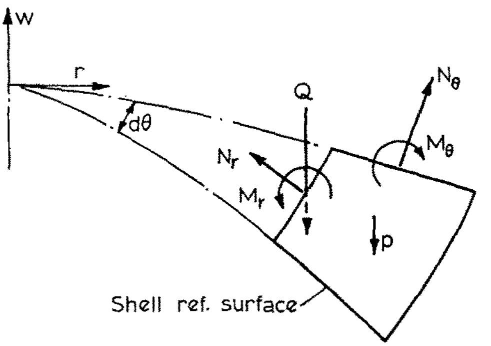


Fig. 2. Positive stress resultants.

(c) *The kinematic equations are*

$$\lambda(2n_r - n_\theta) = U' + W \tag{8}$$

$$\lambda(2n_\theta - n_r) = \frac{1}{\xi} U + W \tag{9}$$

$$\lambda(2m_r - m_\theta) = h\beta' \tag{10}$$

$$\lambda(2m_\theta - m_r) = \frac{h}{\xi}\beta \tag{11}$$

$$\lambda 2\sqrt{3}\, q = W' + \beta. \tag{12}$$

In the above equations the transverse normal shear stress resultant $q$ has been included. The effect of $q$ on the analysis and the resulting yield point loads has been investigated in[4].

(d) *The boundary conditions are at the shell edge* $\xi = c$:

$$W = 0, m_r = 0, U = 0 \tag{13}$$

at the shell center $\xi = 0$:

$$U = 0, \beta = 0, W = W_0 \tag{14}$$

and from symmetry at $\xi = 0$:

$$n_r = n_\theta;\ m_r = m_\theta, q = 0. \tag{15}$$

## METHOD OF SOLUTION

A numerical approach is suggested for equations (4–12). For this purpose the equations are rewritten to make them suitable for numerical integration.

Equation (7) is solved for

$$m_\theta = \frac{m_r}{2} - [-\tfrac{3}{4}m_r^2 - q^2 + 1 - n_r^2 - n_\theta^2 + n_r n_\theta]^{1/2}. \tag{16}$$

This value for $m_\theta$ is inserted into equation (6), which becomes

$$m_r' = \frac{q}{\sqrt{3}h} - \frac{m_r}{2\xi} - \frac{1}{\xi}[1 - \tfrac{3}{4}m_r^2 - q^2 + 1 - n_r^2 - n_\theta^2 + n_r n_\theta]^{1/2}. \tag{17}$$

Equations (8–12) yield, after elimination of the flow parameter $\lambda$,

$$\beta' = \frac{\beta}{\xi}\frac{2m_r - m_\theta}{2m_\theta - m} \tag{18}$$

$$W' = \beta\left[\frac{2\sqrt{3}\,qh}{\xi(2m_\theta - m_r)} - 1\right] \tag{19}$$

$$U' = \frac{h\beta}{\xi}\left[\frac{2n_r - n_\theta}{2m_\theta - m_r}\right] - W \tag{20}$$

$$U = \xi\left[\frac{h\beta}{\xi}\frac{2n_\theta - n_r}{2m_\theta - m_r} - W\right]. \tag{20a}$$

The scheme is as follows:

(a) Assume values for the shell parameters $h$ and $c$ and the load $P$.

(b) The stress resultant profile for $n_r$ is taken as

$$n_r = A\sin\frac{\pi\xi}{c} + B(\xi - c) + D \tag{21}$$

where $A$, $B$ and $D$ are arbitrary constants.

This form of the $n_r$ distribution was chosen after trying several other forms which did not give acceptable velocity fields. Equation (21) is also an extension of the stress field obtained in[11].

(c) The value of $n_\theta$ is found from equation (4) and $q$ is calculated from equation (5).

(d) With the 4th order Runge–Kutta method[7] equation (17) is integrated from $m_r$ and subsequently $m_\theta$ is found from equation (16).

(e) Adjust the value of $P$ until the boundary condition $m_r = 0$ is satisfied.
(f) So far in the scheme a lower bound on the exact $P$ has been found. This lower bound can be improved by adjusting the values of the arbitrary constants in equation (21).
(g) To find the velocity field, equations (18–20) are integrated in turn using the Runge–Kutta method.

It will be found that the constants in equation (21) can be adjusted readily until an admissible velocity results. Since the assumed distribution of $n_r$ may however not include the exact $n_r$ distribution, equations (20) and (20a) will in general give different values of the inplane velocity $U$ which are physically admissible. The constants in equation (21) are adjusted until the two calculated values of $U$ are reasonably close and both fulfil the boundary conditions.

In the numerical integration we found that although the $U$ velocity was very sensitive to small changes in the parameters of equation (21), the collapse load $P$ was not sensitive to these changes.

The results for collapse loads obtained were taken as acceptable when changes in equation (21) resulted in a kinematically admissible velocity field and resulted in changes of the collapse loads $P$ of less than 2 per cent.

(h) The end result of this iterative scheme is an admissible stress field, an admissible velocity field and a corresponding approximate collapse load $P$.

The accuracy of the obtained collapse load $P$ depends on the accuracy of the inplane velocity $U$ as obtained in equations (20) and (20a). The results for $P$ given here are estimated to be within $\pm 2$ per cent of the exact solution.

Note that when transverse shear is considered it is necessary to specify two shell parameters independently. Since we are considering relatively thin shells, the reduction of the collapse load due to shear is small[4]. It is therefore permissible, without loss in accuracy, to plot the resulting collapse loads as a function of only one combined shell parameter [e.g. $c/\alpha$). In[4] it has been shown that the influence of transverse shear on collapse load is only of the order of a few percent for the shell thicknesses considered here. The present analysis is however not simplified considerably if shear is neglected. Although transverse shear is of minor importance for the present problem, it may be important to consider shear effects in problems that involve discontinuous loading or variable rigidity shells.

It may be of interest to some readers that the computer time involved in the present solution is quite small. The time used was about 1 sec for each value of $P$ tried, at a certain set of constant values of $A$, $B$ and $C$. The system used was an IBM–360.

## RESULTS AND DISCUSSIONS

### 1. *Restrained shell edge*

The results for limit loads, using the method of analysis outlined, are shown in Fig. 3 as curve $A$. Here also are shown the results obtained in[1] and[8]. The bounds on limit loads obtained in[1] are labelled $B$, $C$ and $D$ and the upper bound solution obtained in[8] is shown as curve $F$. The results of the present analysis, curve $A$, show good agreement with the results of[8], which is an upper bound solution using the Tresca yield criterion.

The results for limit loads can be expressed by simple equations which are useful to the analyst or designer. For the sake of completeness we also give the equations of[1] here.

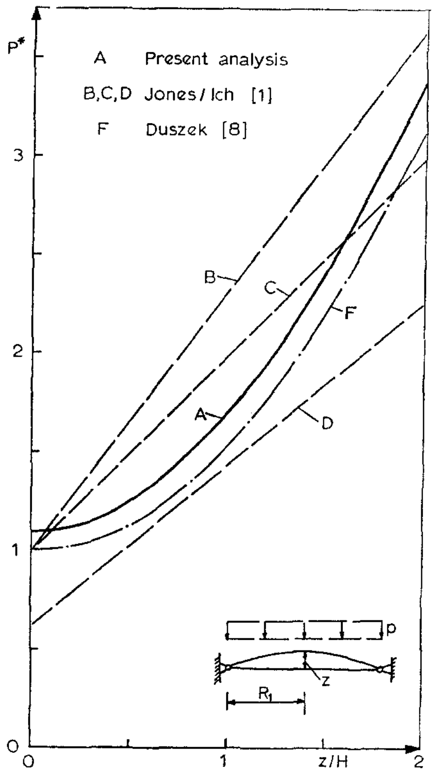


Fig. 3. Limit loads of edge-restrained shallow spherical shells.

The lower bound solution (curve $D$) is expressed by

$$P^* = 0{\cdot}618\left[1 + \frac{4}{3}\frac{z}{H}\right]. \tag{22}$$

The upper bounds are curve $B$

$$P^* = 1 + \frac{4}{3}\frac{z}{H} \tag{23}$$

and curve $C$

$$P^* = 1 + \frac{z}{H}. \tag{24}$$

Curve $F$ represents the upper bound solution of[8]. which can be written as

$$P^* = 1 + \frac{8}{15}\left(\frac{z}{H}\right)^2. \tag{25}$$

The results of the present analysis, curve $A$, can be closely approximated by the equation

$$P^* = 1{\cdot}09 + 0{\cdot}63\left(\frac{z}{H}\right)^{k_1} \tag{26}$$

where the exponent

$$k_1 = 1{\cdot}5, \text{ if } \frac{z}{H} \leq 1{\cdot}0$$

and

$$k_1 = 1{\cdot}85, \text{ if } \frac{z}{H} > 1{\cdot}0.$$

Note that in arriving at equations (25) and (26) terms $\left(\frac{z}{R_1}\right)^2 \ll 1$ have been neglected.

The collapse loads $P^*$ are normalized with respect to the Tresca collapse loads. For the circular plate ($z = 0$) we recover from equation (22) to (25) the Tresca load $P^* = 1$ (or $p\pi R_1^2 = 6\pi M_0$) and from equation (26) we recover the Mises load $P^* = 1{\cdot}09$ (or $p\pi R_1^2 = 6{\cdot}51\pi M_0$)

2. *Unrestrained shell edge*

Since no experimental evidence is available to indicate which of the formulae best predicts the limit load of the edge-restrained shell, it is of interest to briefly examine the case of the unrestrained shell.

Figure 4 shows some available data for the unrestrained shallow shell. Curve $J$ shows the results taken from[4] for which the present method of analysis has been used. The predictions of the theory agree quite well with experimental data ($E1$ to $E3$) taken from[9]. Experimental data $E1$ is for an extremely thick shallow shell whose parameter $\alpha$ lies outside the range given by equation (2).

In[9] shells made of a strain hardening material were tested. The collapse (or limit) pressure was taken as the pressure at the point of intersection of the tangents in the elastic and plastic ranges of the respective pressure-deflection diagrams.

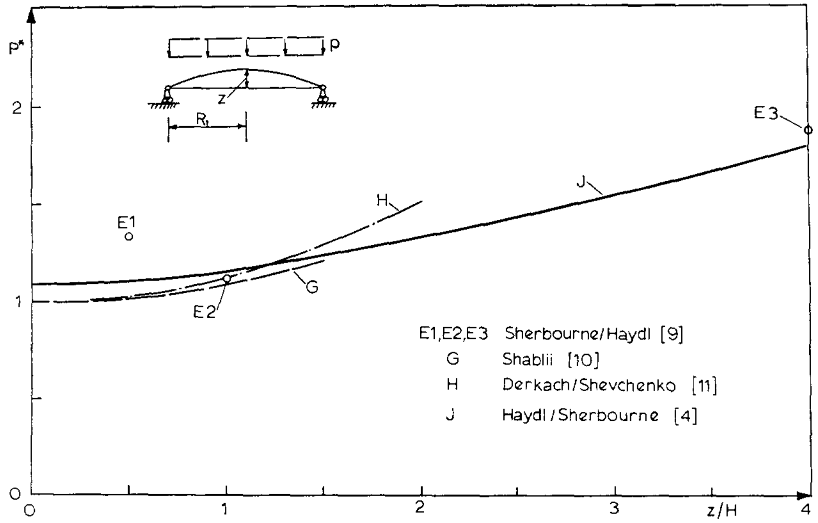


Fig. 4. Limit loads of unrestrained shallow spherical shells.

The predictions of[10] and[11] based on the Tresca criterion are shown as curves $G$ and $H$.

Again we can express the theoretical predictions by simple formulae. Curve $J$ is approximated by

$$P^* = 1{\cdot}09 + 0{\cdot}08 \left(\frac{z}{H}\right)^{3/2}. \tag{27}$$

Curve $G$ can be expressed as

$$P^* = 1 + \frac{4}{45}\left(\frac{z}{H}\right)^2 \tag{28}$$

and curve $H$ as

$$P^* = 1 + \frac{17}{135}\left(\frac{z}{H}\right)^2. \tag{29}$$

In equation (27) and (28) terms $\left(\frac{z}{R_1}\right)^2 \ll 1$ have been neglected.

## SUMMARY AND CONCLUSIONS

Limit load solutions for edge-restrained shallow spherical shells under uniform pressure are obtained. The analysis uses a yield surface which is based on the von Mises yield criterion. A numerical scheme has been outlined and it is shown that the results agree with previous analyses based on the Tresca criterion. To indicate the usefulness of the present analysis the problem of the shell with unrestrained edge is briefly reviewed. Based on this we suggest that the present method of analysis and results for limit loads may more accurately predict the real problem. This can of course only be ascertained when some experimental data becomes available for the edge-restrained shells.

The present method of analysis is an attempt to obtain exact numerical solutions to the equations of plastic shallow shells. The scheme outlined here has the advantage that the solution is approached from the lower bound stand point. The computations of the velocity fields, especially the inplane velocity $U$, do not have to be extremely accurate to obtain collapse loads close to the exact collapse loads. It remains to be shown if the "exact" collapse analyses predict the real collapse of many structural members better than some "approximate" solutions, as the one suggested here.

The numerical scheme outlined has one main disadvantage, i.e. one has to have some insight into the problem such that a reasonable $n_r$ distribution (equation 21) is chosen at the outset of the calculations.

The authors suggest that the present method in principle can be extended to other geometries, loading conditions and boundary conditions, and may be applied if other classical limit analysis approaches fail.

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**Резюме** — В работе приводится расчет предельной нагрузки на идеально пластичные сферические корпусы с заделанными краями подвергнутыми наружному давлению. Численное решение, на основании критерия текучести ван-Майзеса достаточно хорошо соответствовало существующим приблизительным решениям по критерию текучести Треска.